C.S. Peirce Diagrammatic Calculus

Gianluca Caterina
Tufts University
April 20 2022

Outline

- Introduction to Peirce's Existential Graphs.
- Cuts-only graphs: soundness and completeness.
- Alpha graphs; soundness and completeness.

C.S. Peirce (1839-1914)

- Born in Cambridge
- His father was Benjamin Peirce (1809-1880).
- Difficult personal life.
- Contributions to logic, philosophy, semiotic.
- Considered the founder of pragmatism.

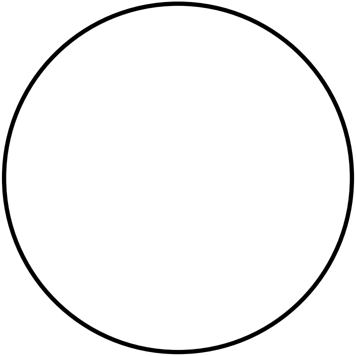
C.S. Peirce (1839-1914)

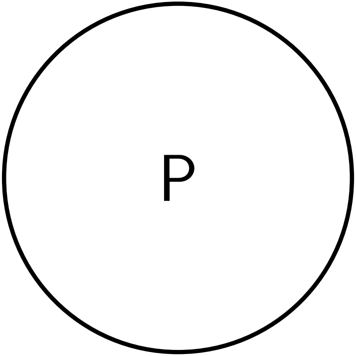
- He defined the notion of *abduction:*

  " The surprising fact, C, is observed;
  But if A were true, C would be a matter of course,
  Hence, there is reason to suspect that A is true."
  (CP 5.189, 1903)

Logic at the end of the 19^{th} century

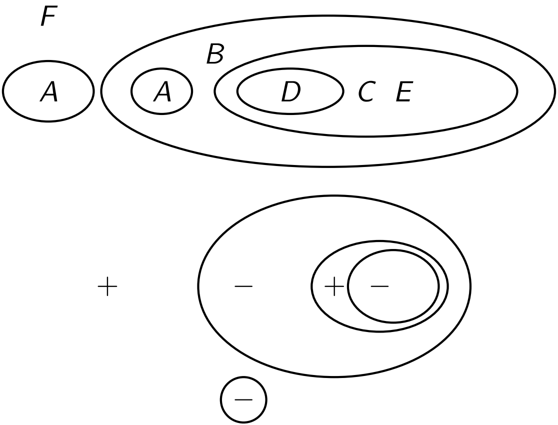
- Logic was far from being formalized in modern terms.
- G. Frege (1848-1925) *Begriffsschrift, a Formula Language, Modeled Upon That of Arithmetic, for Pure Thought* (1879).
- G. Boole (1815-1864) *The Mathematical Analysis of Logic* (1847).
- C. Peirce *On the Algebra of Logic: A contribution to the Philosophy of Notations* (1885).
- Peirce first defined quantifiers as known today.
- He then turned to conceive quantifiers in terms of relations.




P

Peirce's Existential Graphs EG_α: Informal definition

1. The blank sheet (called the Sheet of Assertion or *SA*) is both the site on which graphs are *scribed* and is itself a graph (called the empty graph).
2. A *character* is any reproducible image that is not a *cut* (see below) scribed on part of the *SA*.
3. Characters may be enclosed, along with a local area surrounding them (a neighborhood on the *SA* that may or may not include other characters or figures), by a closed curve called a *cut*. It is convenient to construct these generally as ovals or circles. These cuts may not intersect characters, nor may they intersect one another.

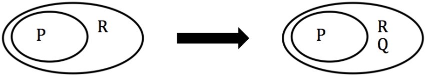

F
A
A
B
D
C E
+
−
+
−
−

EG_α rules

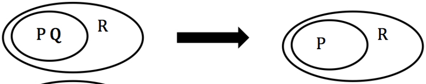
1. (WO) Inscribing any graph on an odd area;
2. (EE) Erasing any graph from an even area;
3. (IT^+) Copying and pasting a subgraph H on any area that is inside the area that contains H and is not part of H itself;
4. (IT^-) Deleting any subgraph that can be obtained using IT^+;
5. Inserting (DC^+) or erasing (DC^-) double cuts around any subgraph of G.

Derivation Rules:

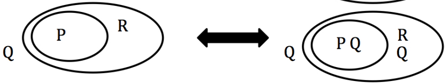
Write Odd:



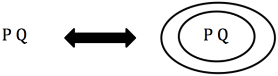
Erase Even:



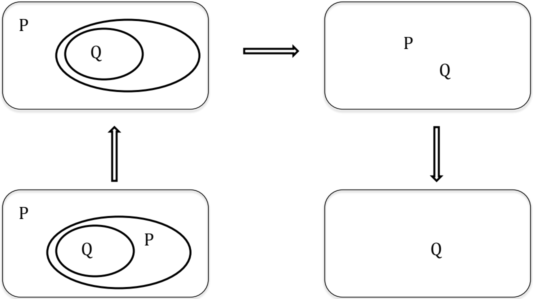
De/Iterate:



Double Cut Write/Erase:



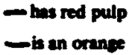
Modus Ponens



Peirce's Existential Graphs: EG_β

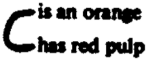
Something has red pulp and something is an orange:

$\exists x\ P(x) \wedge \exists y\ Q(y)$



There is an orange which has red pulp:

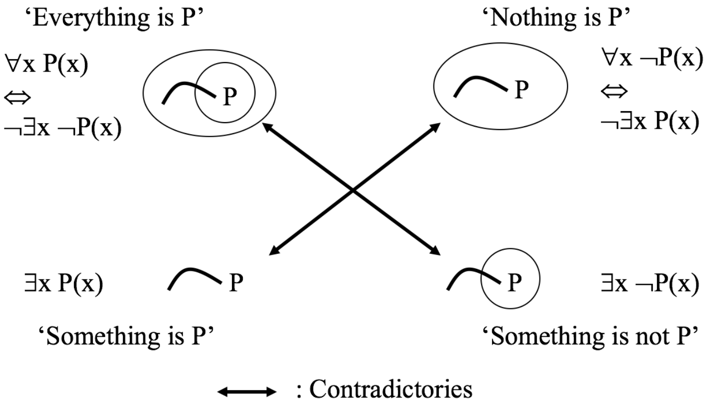
$\exists x\ P(x) \wedge Q(x)$

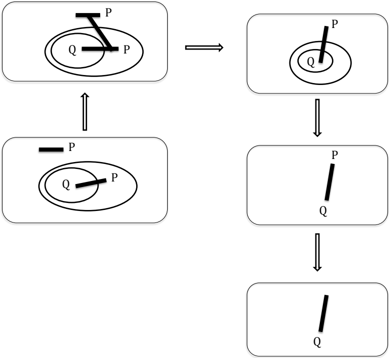


There is something in between two things:

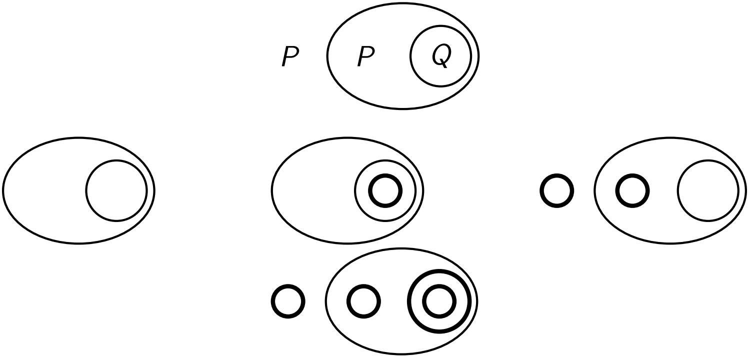
$\exists x \exists y \exists z\ R(x, y, z)$

The Boolean Square of Opposition



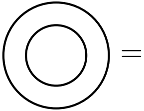

P
Q
P
P
Q
P
Q
P
P
Q
Q

Cuts-only graphs

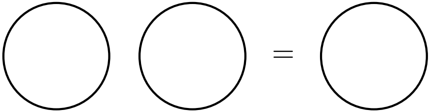


Spencer-Brown

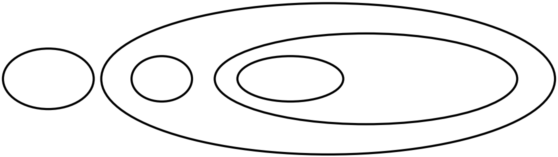
1. *Law of crossing*

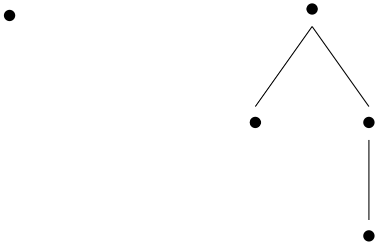


2. *Law of calling*



Cuts-only graphs as trees





A tree-based evaluation algorithm

Procedure

(*i*). *Consider the forest representation F of the graph* G^**.*

(*ii*). *Take any pre-terminal node n in F.*

(*iii*). *Eliminate the node n and all nodes below it, yielding a new forest* F'*.*

(*iv*). *Replace F with* F' *and repeat* (ii) *and* (ii) *until the forest* F' *is either empty or consists of single nodes only (nodes of at most depth* 1*).*

(*v*). *If the forest* F' *is empty, replace it with the empty Sheet of Assertion. If it consists of single nodes only, replace it with a single empty cut.*

A diagrammatic logic

Definition

Let us consider two cuts-only graphs G^* and H^*. We say that G^* entails syntactically H^* in n steps *if and only if there is a sequence of graphs*

$$G^* = G_0^*, G_1^*, \dots G_n^* = H^*$$

such that G_{i+1}^* is derivable from G_i^* using one of the transformation rules, for all $i = 0, \dots, n-1$. We use the notation

$$G^* \vdash^n H^*$$

to indicate that G^* entails syntactically *graph H^* in n steps (in what follows, we may drop the subscript n when not needed).*

Definition

Let us consider two cuts-only graphs G^* *and* H^**. We say that* G^* entails semantically H^* *if and only if*

$$V(G^*) = \quad \Rightarrow V(H^*) =$$

We use the notation

$$G^* \models H^*$$

to indicate that G^* entails semantically *graph* H^**.*

Metatheorems

- Soundness and completeness hold for cuts-only graphs.
- Soundness immediately entails consistency, that is the empty cut cannot be derived by the empty sheet.
- Known proofs of completeness for EG_α use a translation into PL, Henkin-style strategy.
- Idea: lift soundness and consistency for cuts-only to the full EG_α.
- We found two *native* completeness proofs:
  - Reduction to Conjunctive Normal Form.
  - Kalmar's proof for PL (1935).

Native completeness

Let us first recall the Deduction Theorem for EG_α:

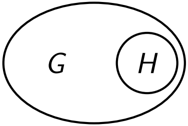
Theorem

Suppose that $G_1, G_2, \ldots, G_n, G$ are graphs such that

$$G_1, G_2, \ldots, G_n, G \vdash H$$

then

$$G_1, G_2, \ldots, G_n \quad \vdash$$



Semantic Entailment

For any alpha graph G let us also define G^{*h} to be the cuts-only graph obtained by replacing each variable X in G with $h(X)$.
We can now define semantic entailment for alpha graphs:

Definition

Let us consider two EG_α graphs G and H. We say that G entails semantically *H if and only if*

$$V(G^{*h}) = \quad \Rightarrow V(H^{*h}) =$$

for all interpretations h.

We use the notation $G \vDash H$ to indicate that G *entails semantically* the graph H.

Fix an interpretation h.

$$\left(G \right)_h = \begin{cases} G & V(G^{*h}) = \\ \textcircled{G} & V(G^{*h}) = \bigcirc \end{cases}$$

To simplify the notation, we fix h and will drop it from the notation.

Lemma

For any graph G with variable letters $x_1, x_2, \ldots, x_n$ and any interpretation h, we have that

$$\left(x_1 \right), \left(x_2 \right) \ldots, \left(x_n \right) \vdash \left(G \right)$$

First notice that, by construction

that it, the cuts-only graph  evaluates to the empty sheet for any G, and therefore it is provable: that is, it is derivable from the empty sheet in a finite number of steps. But then from the graphs

we can derive

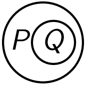
The structure of variables defined by $(G^\star)$ is the same as that of (G) except that all the variables that evaluate to true are replaced by an empty sheet placeholder, and those that evaluate to false are replaced by an empty cut placeholder. But now we can draw from our stash of variables that are written on the empty sheet and just iterate inside $(G^\star)$. $\square$

Example

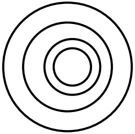
As a concrete example, consider the graph G representing the classical implication $P \Rightarrow Q$ and consider the interpretation $h(P) = \quad$ and $h(Q) = \bigcirc$. The claim of the lemma is then that

$$P, \ \textcircled{Q} \vdash \textcircled{G}$$

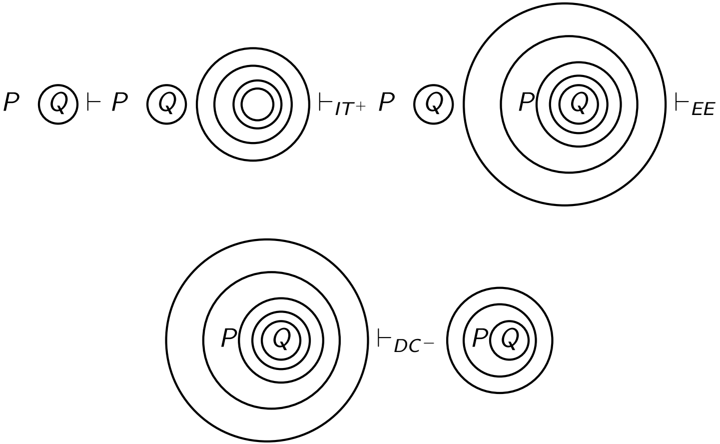
where G is the graph



We know that, by construction, $(G^\star)$ (dashed) is provable. Since the latter is equal to $\textcircled{G^\star}$, that means that the following graph is provable:



But then [1]



[1]In the sequence below, for the sake of brevity the two iterations for P and Q have been collapsed to a single one.

Completeness

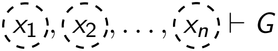
Completeness follows using a standard argument of variables elimination. In EG_α however, using iteration and the dual role of the sep as an operator and operand, the proof becomes very short and streamlined, and we thought it would be useful for the reader to include the details of it below.

Theorem

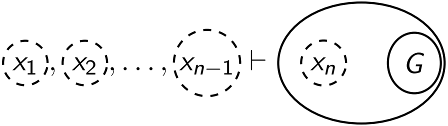
For any $G \in EG_\alpha$ we have that

$$\models G \quad \Rightarrow \vdash G$$

Proof: Let us assume that $var(G) = x_1, x_2, \ldots, x_n$. Since G is a tautology, then $\textcircled{G} = G$ and by the previous lemma we have that



By the deduction theorem we have that



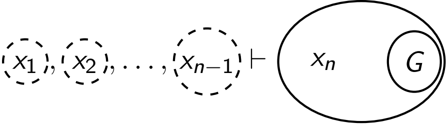
Let us pick two interpretations h_1 and h_2 such that they differ only on x_n, that is, such that

$$h_1(x_i) = h_2(x_i),\ i = 1, \ldots, n-1$$

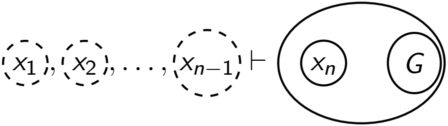
and

$$h_1(x_n) = \quad , h_2(x_n) = \bigcirc$$

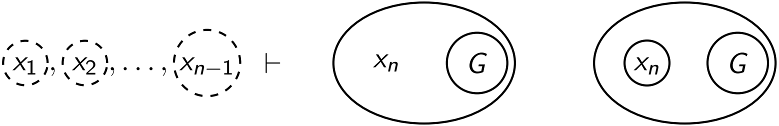
By the previous lemma we have that



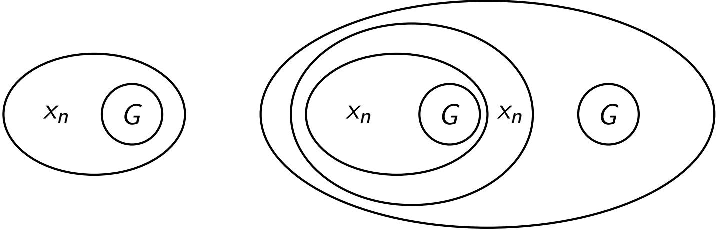
and



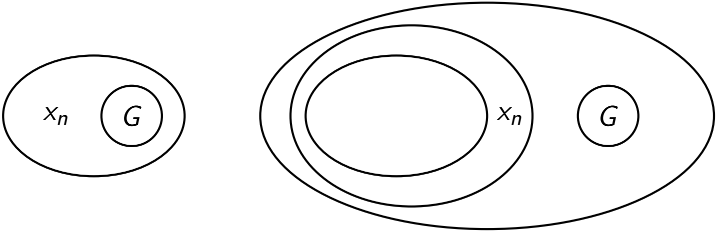
From the above we infer that



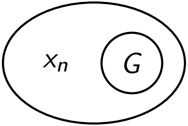
$\vdash_{IT^{+}}$

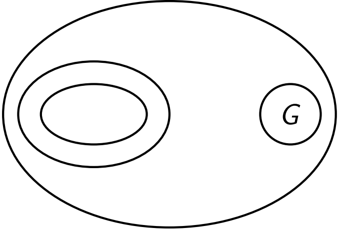


$\vdash_{IT^{-}_{\times 2}}$

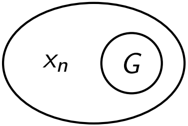


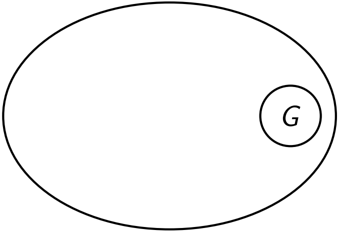
$\vdash_{EE}$



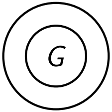


$\vdash_{DC-}$





$\vdash_{EE}$



$\vdash_{DC-}$ G